RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College under University of Calcutta)

B.A./B.SC. THIRD SEMESTER EXAMINATION, DECEMBER 2013

SECOND YEAR

MATHEMATICS (Honours)

Date : 14/12/2013

Time : 11 am – 3 pm

Paper : III

Full Marks : 100

[Use Separate Answer Scripts for each group]

Group – A

(*Answer **any five** of the following*)

1. a) Let V and W be two finite dimensional vector spaces over the same field F and $T: V \to W$ be a linear transformation. If T is injective then prove that the image of a basis in V forms a basis in W. [3]

   b) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x,y,z) = (3x+z, -2x+y, -x+y+4z), \forall (x,y,z) \in \mathbb{R}^3$. Examine whether T is invertible. [3]

   c) Let V be a finite dimensional vector space over a field F and W be a subspace of V. Prove that $\dim V/W = \dim V - \dim W$. [4]

2. a) Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ belonging to the group $G = \left\{ \begin{bmatrix} x & x \\ x & x \end{bmatrix} : x \in \mathbb{R}^* \right\}$ under matrix multiplication. Find the four fundamental subspaces of A and show them geometrically in $\mathbb{R}^2$. [Four fundamental subspaces of A are, rowspace of A, columnspace of A, $\{x \in \mathbb{R}^2, Ax = 0\}$ and $\{y \in \mathbb{R}^2; A^t y = 0\}$, $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$] [5]

   b) The matrix representation of a linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ is $\begin{pmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$ relative to the standard ordered basis of $\mathbb{R}^3$. Find the explicit representation of T and the matrix of T relative to the ordered basis $\{(0,1,1),(1,0,1),(1,1,0)\}$ of $\mathbb{R}^3$. [5]

3. a) Define orthogonal subset of an inner product space. Let V be an inner product space over F ($\mathbb{R}$ or $\mathbb{C}$) and $\alpha, \beta \in V$. Prove that $|<\alpha,\beta>| \le \|\alpha\| \, \|\beta\|$ [1+4]

   b) Consider the system of linear equations

   $px_1 + x_2 + x_3 = 4$

   $x_1 + tx_2 + x_3 = 3$

   $x_1 + 2tx_2 + x_3 = 4$

   Discuss its consistency with respect to p and t and give solution, whenever it may exist. [5]

4. a) Let A and B be two m×n matrices over a field F. Then prove that rank of (A+B) $\le$ rank of A + rank of B. [3]

   b) Let $V = P_2(\mathbb{R})$ with standard ordered basis $\beta = \{1, x, x^2\}$. Considering the inner product $< f(x), g(x) > = \int_{-1}^{1} f(t)g(t)dt$ on $P_2(\mathbb{R})$, use Gram-Schmidt process to replace β by an orthogonal basis and then convert it to the corresponding orthonormal basis. [5+2]

5. a) State Cayley-Hamilton theorem. Use Cayley-Hamilton theorem to find A^{-1}, where $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{pmatrix}$ [1+3]

   b) Prove that similar matrices have the same eigen values. [2]

c) Find the eigenvalues of the matrix $A=\begin{pmatrix}1 & -1 & 2\\ 2 & -2 & 4\\ 3 & -3 & 6\end{pmatrix}$. Also find the eigen vectors of A corresponding to the 2-fold eigen value of A. [2+2]

6. a) Find the matrix P such that $P^{-1}AP$ is a diagonal matrix, where $A=\begin{pmatrix}1 & 1 & -2\\ -1 & 2 & 1\\ 0 & 1 & -1\end{pmatrix}$. [5]

b) Prove that the eigen values of a Hermitian matrix are all real. [3]

c) Let a square matrix A be orthogonally diagonalisable. Prove that A is a symmetric matrix. [2]

7. a) Convert $f(x_1,x_2,x_3)=2x_1x_2+4x_1x_3$ to canonical from by congruent transformation and provide the coordinate transformation x = Cy used there. [3+2]

b) Define unitary operator on an inner product space. Let V be an n-dimensional inner product space over $F(\mathbb{R} \text{ or } \mathbb{C})$ and T be a linear operator on V. Prove that T is unitary if an only if $\|T(v)\|=\|v\|$ for all $v\in V$. [1+4]

8. a) Let V be a finite dimensional inner product space over a field F $(\mathbb{R} \text{ or } \mathbb{C})$ and T be a linear operator on V. Show that there exist self adjoint operators $T_1, T_2 \in L(V,V)$ such that $T=T_1+iT_2$. Moreover show that T is normal if and only if $T_1T_2=T_2T_1$. [4]

b) Let V be a finite dimensional inner product space over F and let g be a linear functional on V. Then prove that there exists a unique vector $y\in V$ such that g(x) = <x,y> for all $x\in V$.

Show further that y belongs to the orthogonal compliment of the null space of g. [4+2]

Group – B

Answer **any four** questions from Q.No 9-14 :

9. Find the distance of the point (3,8,2) from the line $\frac{x-1}{2}=\frac{y-3}{4}=\frac{z-2}{3}$ measured parallel to the plane $3x+2y-2z+5=0$. [5]

10. Show that the equation of the plane containing the straight line $\frac{y}{b}+\frac{z}{c}=1, x=0$ and parallel to the straight line $\frac{x}{a}-\frac{z}{c}=1,\ y=0$ is $\frac{x}{a}-\frac{y}{b}-\frac{z}{c}+1=0$ and if 2d be the shortest distance between the lines, then prove that $\frac{1}{d^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$. [5]

11. If $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ represents one of the set of three mutually perpendicular generators of the cone $5yz-8zx-3xy=0$, find the equations of the other two. [3+2]

12. Prove that the enveloping cylinder of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ whose generators are parallel to the lines $\frac{x}{0}=\frac{y}{\pm\sqrt{a^2-b^2}}=\frac{z}{c}$ meet the plane z = 0 in circles. [5]

13. Show that perpendiculars from the origin to the generators of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ lie on the surface $\frac{a^2(b^2+c^2)^2}{x^2}+\frac{b^2(c^2+a^2)^2}{y^2}=\frac{c^2(a^2-b^2)^2}{z^2}$. [5]

14. Reduce the equation $4x^2 - y^2 - z^2 + 2yz - 8x - 4y + 8z - 2 = 0$ to its canonical form and hence classify the conicoid represented by it. [4+1]

Answer **any two** questions from Q.No 15-17 :

15. a) A particle describes a path which is nearly a circle about a centre of force $(= \mu u^n)$ at its centre; find the condition that this may be a stable motion. [7]

b) Two equal particles of mass m are attached to the two ends of a weightless elastic string of natural length ℓ and modulus of elasticity λ; the string lies on a smooth horizontal table perpendicular to an edge with one particle just hanging over. Show that the other particle will pass over the edge at the end of time t given by the equation $2\ell + \frac{mg\ell}{\lambda}\sin^2\sqrt{\frac{\lambda}{2m\ell}}t = \frac{1}{2}gt^2$. [8]

16. a) A particle moves with an acceleration which is always towards, and equal to μ divided by the distance from a fixed point O. If it starts from rest at a distance 'a' from O, show that it will arrive at O in time $a\sqrt{\frac{\pi}{2\mu}}$. [5]

b) A particle of unit mass is projected with a velocity V at an inclination α above the horizon in a medium whose resistance is K times the velocity of the particle. Show that the direction of its velocity will make an angle $\frac{\alpha}{2}$ above the horizon after a time $\frac{1}{K}\log\left(1 + \frac{KV}{g}\tan\frac{\alpha}{2}\right)$. [5]

c) Establish the formula $\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2}$ for the motion of a particle describing a central orbit under an attraction P per unit mass, the symbols having their usual meanings. [5]

17. a) Find the tangential and normal components of velocity and acceleration of a particle moving in a plane. [6]

b) A particle, subject to a central force per unit mass equal to $\mu\{2(a^2 + b^2)u^5 - 3a^2b^2u^7\}$, is projected at a distance 'a' with a velocity $\frac{\sqrt{\mu}}{a}$ in a direction at right angles to the initial distance; show that the path is the curve $r^2 = a^2\cos^2\theta + b^2\sin^2\theta$. [6]

c) A particle falls under constant gravity through a distance x, starting from rest. A small resistance per unit mass, equal to K times the square of the velocity acts on the particle. Show that the kinetic energy acquired is approximately mgx (1 – Kx), neglecting higher powers of Kx. [3]